

Thin static charged dust Majumdar–Papapetrou shells with high symmetry in $D \geq 4$.

MARTIN ČERMÁK^{a,b} and MARTIN ZOUHAR^{c,b}

^aInstitute of Physics of Materials,
Žitkova 22, CZ 616 62 Brno, Czech Republic

^bDepartment of Physical Electronics,
Faculty of Science, Masaryk University Brno,
Kotlářská 2, Brno, Czech Republic

^cCEITEC - Central European Institute of Technology,
c/o Masaryk University Brno,
Žerotínovo nám. 9, CZ 601 77 Brno, Czech Republic

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Abstract

We present a systematical study of static $D \geq 4$ space-times of high symmetry with the matter source being a thin charged dust hypersurface shell. The shell manifold is assumed to have the following structure $\mathbb{S}_\beta \times \mathbb{R}^{D-2-\beta}$, $\beta \in \{0, \dots, D-2\}$ is dimension of a sphere \mathbb{S}_β . In case of $\beta = 0$, we assume that there are two parallel hyper-plane shells instead of only one.

The space-time has Majumdar–Papapetrou form and it inherits the symmetries of the shell manifold – it is invariant under both rotations of the \mathbb{S}_β and translations along $\mathbb{R}^{D-2-\beta}$.

We find a general solution to the Einstein–Maxwell equations with a given shell. Then, we examine some flat interior solutions with special attention paid to $D = 4$. A connection to $D = 4$ non-relativistic theory is pointed out. We also comment on a straightforward generalisation to the case of Kastor–Traschen space-time, i.e. adding a non-negative cosmological constant to the charged dust matter source.

Keywords

Majumdar–Papapetrou Kastor–Traschen higher dimensional thin shell charged dust general relativity.

1 Introduction

Finding exact solution to Einstein equations is a difficult task in general. The equations of motion simplify if the sought solution has a symmetry, therefore solutions of high symmetry are relatively easy to study and hence quite well known.

Some modern physical models and theories abandon the assumption that the space-time we live in has four dimensions. The extension to higher dimensions can be motivated by unification attempts, such as Kaluza–Klein (unification of gravity and gauge fields) [1] and string theory (candidate for a quantum gravity theory), or brane-world model, where our world is a brane embedded in a higher dimensional total space-time called bulk (which may explain why gravity, "leaking" to the bulk, is weaker than other interactions that are trapped on the brane) [2].

The above mentioned models and theories give a physical motivation to examine higher dimensional problems that can be interesting mathematical problem by itself.

Let us present a reason to study shells. Potential (both electrostatic and gravitational) of point like and line like sources diverges at the source, probably due to "too singular" nature of the source density. To avoid the singular behavior at the point and line matter sources, it is possible to modify the equations for the field, e.g. consider non-linear electrodynamics [3] that can remove singularities in the field strength tensor $F_{\alpha\beta}$, or modify the matter source distribution, e.g. by considering shells. We shall see that the potential at the shells considered in this work has a finite value.

In $D = 4$ space-time, some examples of the highly symmetric solutions are static spherical, cylindrical and plane spatial symmetry solutions to the Einstein-Maxwell equations, reviewed e.g. in [4]. Particular examples of such solutions are those of thin charged massive shells studied e.g. in [5, 6].

In case of a higher dimensional space-time, the above mentioned three spatial symmetries can be generalised in more ways, hence the considered shell manifold $\mathcal{M} = \mathbb{S}_\beta \times \mathbb{R}^{D-2-\beta}$, see (4.1).

The Majumdar–Papapetrou form metric has already been studied in [7, 8] among others. When compared to [7], that examines a cloud of charged dust, we study a general dimension case and charged dust shells only and we present exact solutions in all the cases of the symmetries, classified by the value of parameter β , considered. This distinguishes our article also from reference [5] the author of which extends his work in [7] by adding rather general $D = 4$ shells considerations with only one exact shell solution in the case of spherical symmetry.

The work [8] comments on some general properties of a higher dimensional space-time of Majumdar–Papapetrou type with proper derivation of the metric form from a quite general setting. A reader curious about the origin of metric in equation (3.1) is referred there though he or she will find no exact solution there.

Moreover, the generalisation to Kastor–Traschen type space-time, missing in [5, 7, 8], is discussed in our article in $D \geq 4$ which is, in some way, an extension of the $D = 4$ treatment in [6].

The shell separates the space-time into two regions that may be called exterior region and interior region¹. The two regions have to be matched across the shell. The metric is continuous across the surface but a quantity related to its first derivative need not be. In general, the first derivative exhibits a jump across the shell. The jump is related to the shell matter source.

We integrate the equations of motion allowing distributions (such as Dirac and Heaviside "functions") enter the expressions for metric components. This allows us to write each exterior and interior solution component in a single formula.

In a special case, the interior space-time region is flat and the exterior is a curved solution of Einstein-Maxwell equations.

If we demand the solution to be non-collapsing, then we have to enforce that repulsive electrostatic forces are in balance with the attractive gravity forces.

¹This terminology is a bit cumbersome if the shell is not a boundary to a compact volume.

Thus obtained balance condition will be examined in both general relativity and non-relativistic physics in this article.

In this work, we consider standard Einstein–Hilbert general relativity with Levi–Civita connection, the signature is mostly minus, see e.g. [9, 10].

Greek indices (run from 0 to $D - 1$) are used for space-time coordinates and Latin indices (run from 1 to $D - 1$) will be used for spatial coordinates. $x^0 = ct$. Partial derivative is denoted by comma, covariant derivative by semicolon.

We shall explicitly write out parameters of the solutions, including universal constants (light velocity c , Newton gravitational constant κ and permittivity of vacuum ε_0).

Outline of this article is: We give some general comments on the equations of motion in the case of matter sources of interest in Section 2. The solutions have the Majumdar–Papapetrou metric form discussed in Section 3. We solve the Einstein–Maxwell equations for the Majumdar–Papapetrou metric for the shells considered in Section 4. Consequently, some sub-cases corresponding to $D \in \{4, 5\}$, and subject to additional conditions, are discussed in detail in Section 5. Then, the non-relativistic approximation follows in Section 6. Consequently, we briefly discuss generalisation to Kastor–Traschen space-time with positive cosmological constant in Section 7.

2 General notes on equations of motion

The field equations for metric, the Einstein equations, are

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R \equiv G_{\alpha\beta} = KT_{\alpha\beta}, \quad (2.1)$$

where K is coupling constant. $K = \frac{8\pi\kappa}{c^4}$ in a four dimensional space-time.

The total stress-energy-momentum tensor of a charged dust consist of two parts - the dust and the electro-magnetic field contributions.

$$T_{\alpha\beta} = \rho_m c^2 u_\alpha u_\beta \frac{ds}{dx^0} + \varepsilon_0 c^2 \left[\frac{1}{4} F^{\gamma\delta} F_{\gamma\delta} g_{\alpha\beta} - F_\alpha{}^\delta F_{\beta\delta} \right]. \quad (2.2)$$

First part represents the dust and the second represents the electro-magnetic field. The field strength tensor $F_{\alpha\beta}$ is related to the gauge potential A_α as $F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu}$.

Since we seek for a static solution, we pick up a coordinate system in which the dust is at rest. As a result, velocity has only one non-zero component so that $u_\alpha = \sqrt{g_{00}}\delta_\alpha^0$ and derivative of the line element s by x^0 is $\frac{ds}{dx^0} = \sqrt{g_{00}}$.

Volume mass density of the shells is mass surface density multiplied by the delta function $\rho_m = \mu\delta(r - r_0)$, where μ is constant mass density on the shell, $\delta(r - r_0)$ is the delta function, r is coordinate that is constant on the shell.

The only non-zero component of the dust (i.e. massive part of the total) stress-energy-momentum tensor is

$$T_{m00} = \mu c^2 g_{00}^{\frac{3}{2}} \delta(r - r_0). \quad (2.3)$$

Volume charge density is $\rho_e = \frac{\sigma\delta(r-r_0)}{\sqrt{-g_{rr}}}$, where σ is surface charge density and g_{rr} is rr component of the metric tensor. We shall use adapted coordinate system in which $r = x^1$.

The equation of motion for the electromagnetic field relates the strength tensor $F_{\alpha\beta}$ to electric current j^α as

$$F^{\alpha\beta}{}_{;\alpha} = \frac{1}{\sqrt{|g|}} \left(\sqrt{|g|} F^{\alpha\beta} \right)_{,\alpha} = \frac{j_e^\beta}{\varepsilon_0 c^2}, \quad (2.4)$$

where we have used identity relating covariant divergence of an anti-symmetric tensor with ordinary divergence.

If the source is at rest, then the current has only one non-zero component given by $j_e^0 = \frac{c\rho_e}{\sqrt{g_{00}}}$.

Charge can be calculated using $dQ = \rho_e \sqrt{\frac{|g|}{g_{00}}} dV$, where $dV = \prod_{i=1}^{D-1} dx^i$ and g is determinant of the metric tensor.

Mass can be calculated in a similar spirit as $dm = \rho_m \sqrt{|g|} dV$.

Using the expression for the charge current components, the equation (2.4) becomes

$$\left(\sqrt{|g|} F^{\alpha 0} \right)_{,\alpha} = \frac{\sqrt{|g|}}{\sqrt{g_{00}}} \frac{\rho_e}{\varepsilon_0 c}, \quad \left(\sqrt{|g|} F^{\alpha i} \right)_{,\alpha} = 0. \quad (2.5)$$

3 Einstein equations for Majumdar–Papapetrou metric

Majumdar–Papapetrou metric describes space-time of charged matter with attractive gravitational force balanced by repulsive electrostatic force. The Majumdar–Papapetrou metric can also describe static charged dust shells.

A general form of the metric tensor is given by [8, 11]

$$ds^2 = G^{-2} (c dt)^2 - G^{2/(D-3)} \gamma_{ij} dx^i dx^j, \quad (3.1)$$

where γ_{ij} is a flat $(D - 1)$ -dimensional metric, i.e.

$$\gamma_{ij} dx^i dx^j = \delta_{kl} d\tilde{x}^k d\tilde{x}^l = (d\tilde{x})^2,$$

where \tilde{x}^k are Cartesian coordinates in the above formula.

Consider static Majumdar–Papapetrou D -dimensional metric in Cartesian coordinate system. We drop the tilde over x in the following expressions.

The corresponding Einstein tensor has following non-zero components

$$G_{00}^{\text{MP}} = \frac{1}{2} \frac{D-2}{D-3} \frac{(\nabla G)^2 - 2G\Delta G}{G^{4+2/(D-3)}}, \quad G_{ij}^{\text{MP}} = \frac{1}{2} \frac{D-2}{D-3} \frac{(\nabla G)^2 \delta_{ij} - 2G_{,i} G_{,j}}{G^2}, \quad (3.2)$$

where we have defined quantities related to flat space operators – square of a scalar gradient and Laplace operator – as

$$(\nabla G)^2 = \gamma^{ij} G_{,i} G_{,j}, \quad \Delta G = \frac{1}{\sqrt{|\gamma|}} \left(\sqrt{|\gamma|} \gamma^{ij} G_{,i} \right)_{,j}, \quad |\gamma| \equiv |\det(\gamma_{ij})|. \quad (3.3)$$

The electrostatic field is given by

$$A_\alpha = \frac{\phi_e(x^i) G^{-2}}{c} \delta_\alpha^0, \quad F_{\alpha\beta} = \delta_\beta^0 A_{0,\alpha} - \delta_\alpha^0 A_{0,\beta}, \quad F^{i0} = -\delta^{ij} \frac{(\phi_e G^{-2})_{,j}}{c} G^{2\frac{D-4}{D-3}}, \quad (3.4)$$

where $\phi_e \equiv cA^0$ is electrostatic potential.

The non-zero components of the EM metric stress-energy-momentum tensor are written in terms of A_0 – instead of the function ϕ_e – for the sake of brevity

$$T_{e00} = \frac{\varepsilon_0 c^2}{2} \frac{(\nabla A_0)^2}{G^{2/(D-3)}}, \quad T_{eij} = \frac{\varepsilon_0 c^2}{2} [(\nabla A_0)^2 \delta_{ij} - 2A_{0,i} A_{0,j}] G^2, \quad (3.5)$$

where $(\nabla A_0)^2 = \gamma^{ij} A_{0,i} A_{0,j}$ was introduced in the same way as $(\nabla G)^2$.

It is easy to show, using (2.3), that the only non-zero component of the dust stress-energy-momentum tensor becomes

$$T_{m00} = \frac{c^2 \rho_m}{G^3}. \quad (3.6)$$

Einstein equations relate the metric function G , electrostatic potential ϕ_e and density of mass ρ_m in the following way [12]

$$\frac{1}{G} = C \mp \sqrt{K \varepsilon_0} \sqrt{\frac{D-3}{D-2}} \frac{\phi_e}{G^2}, \quad \Delta G = - \frac{D-3}{D-2} K c^2 G^{\frac{2}{D-3}} \rho_m, \quad (3.7)$$

where Δ is the flat space Laplace operator, C is a constant of integration and K is coupling constant. The Einstein equation have so far allowed for an ambiguity in sign relating G and ϕ_e , hence the \mp .

The mathematical solution for ϕ_e in terms of the metric function G presented in (3.7) contains an integration constant C which we shall put equal zero in order to obtain a good correspondence with non-relativistic physics, then $G = \mp \sqrt{K\varepsilon_0} \sqrt{\frac{D-3}{D-2}} \phi_e$.

The EM field equations of motion (2.5), together with both (3.4) and (3.7), lead to

$$\begin{aligned}
G^{\frac{D-1}{D-3}} \frac{\rho_e}{\varepsilon_0 c} &= \frac{\sqrt{|g|}}{\sqrt{g_{00}}} \frac{\rho_e}{\varepsilon_0 c} = \left(\sqrt{|g|} F^{\alpha 0} \right)_{,\alpha} = \left(\sqrt{|g|} F^{i0} \right)_{,i} = \left(\sqrt{|g|} F_{j0} g^{00} g^{ji} \right)_{,i} \\
&= \left[-\sqrt{|g|} g^{00} g^{ji} \frac{(\phi_e G^{-2})_{,j}}{c} \right]_{,i} = \left[-G^2 \delta^{ij} \frac{(\phi_e G^{-2})_{,j}}{c} \right]_{,i} \\
&= \mp \sqrt{\frac{D-2}{D-3}} \frac{1}{c \sqrt{K\varepsilon_0}} \delta^{ij} G_{,ji} = \mp \sqrt{\frac{D-2}{D-3}} \frac{1}{c \sqrt{K\varepsilon_0}} \Delta G \\
&= \pm c \sqrt{\frac{D-3}{D-2}} \sqrt{\frac{K}{\varepsilon_0}} G^{\frac{2}{D-3}} \rho_m.
\end{aligned}$$

Thus it implies balance condition for mass and charge densities

$$c^2 \sqrt{K\varepsilon_0 \frac{D-3}{D-2}} \rho_m = |\rho_e| G. \quad (3.8)$$

We have written an absolute value of charge density to ensure mass density is positive. This corresponds to a certain choice of the \pm factor, i.e. the sign ambiguity is removed.

The condition (3.8) can be written using the total quantities by using integral relations

$$M = \int_V \sqrt{|g|} \rho_m dV, \quad Q = \int_V \sqrt{\frac{|g|}{g_{00}}} \rho_e dV, \quad (3.9)$$

where the integration domain V must contain the shell, i.e. $\mathcal{M} \subset V$.

The balance condition becomes

$$M c^2 \sqrt{K\varepsilon_0 \frac{D-3}{D-2}} = |Q|. \quad (3.10)$$

We put $K = \frac{D-2}{D-3} \frac{4\pi\kappa}{c^4}$ ². Then the balance condition (3.10) is independent of number of space-time dimensions

$$M \sqrt{4\pi\kappa\varepsilon_0} = |Q|. \quad (3.11)$$

We have found general form of the solution – the electrostatic potential ϕ_e as a function of the metric function G , equation (3.7), a formula relating mass and charge densities of a charged dust, equation (3.8). Moreover, the equation (3.7) also includes the relation between G and mass density ρ_m .

Thus it suffices to prescribe e.g. the mass density ρ_m , solve the second relation in equation (3.7) with respect to G and the space-time is fully determined. This will be done in the next section.

² Two approaches to the coupling constants can be found in the literature. The first assumes that the Poisson equation, obtained in non-relativistic limit, retains the same form in higher dimensions. The second corrects the coupling constant in the Poisson equation so that it explicitly depends on the space-time dimension [13].

4 Thin shell solution of high symmetry in $D \geq 4$

We shall consider charged dust shells being hypersurfaces \mathcal{M} in the Euclidean space $(\mathbb{R}^{D-1}, \gamma)$.

$$\mathbb{R}^{D-1} \supset \mathcal{M} = \mathbb{S}_\beta \times \mathbb{R}^{D-2-\beta}, \dim \mathcal{M} = D-2, \beta \in \{0, 1, \dots, D-2\}. \quad (4.1)$$

Since the shells considered are of high symmetry, we shall work in a coordinate system adapted to the symmetries instead of the Cartesian one used in the previous section. We are going to denote $r = x^1$ a coordinate describing distance from the shell in such an adapted coordinate system.

We may express it as a function of spatial Cartesian coordinates x^i in the following way $r = \sqrt{\sum_{i=1}^{\beta+1} (x^i)^2}$, where β is determined by the choice of shell manifold (4.1).

The choice of \mathcal{M} affects adapted coordinate system. The line element corresponding to the metric γ_{ij} in (3.1) is

$$\gamma_{ij} dx^i dx^j = \underbrace{(d\vec{x})^2}_{\mathbb{R}^{D-1}} = \underbrace{dr^2 + r^2 d\Omega_\beta^2}_{\mathbb{R} \times \mathbb{S}_\beta} + \underbrace{\delta_{AB} dx^A dx^B}_{\mathbb{R}^{D-2-\beta}}, \quad (4.2)$$

where the coordinates are labelled as follows

$$x^1 = r, \{x^\Gamma = \theta_{\Gamma-1}\}_{\Gamma=2}^{\beta+1}, \theta_\beta = \varphi, \{x^A = z_{A-\beta-1}\}_{A=\beta+2}^{D-1}. \quad (4.3)$$

$d\Omega_\beta^2$ denotes volume element of a β -dimensional sphere in equation (4.2). It can be written as

$$d\Omega_\beta^2 = \sum_{\Gamma=1}^{\beta-1} \left[\prod_{\Upsilon=1}^{\Gamma} \sin^2(\theta_\Upsilon) \right] (d\theta_{\Gamma+1})^2 + d\theta_1^2.$$

One can easily find that

$$\sqrt{|\gamma|} = r^\beta f(\theta), \quad f_{,r} = 0 \quad (4.4)$$

holds in the adapted coordinate system (4.3).

We assume the space-time has following symmetries – static, spherical symmetry of \mathbb{S}_β and translational symmetry along the $\mathbb{R}^{D-2-\beta}$ factor. These symmetries correspond to following set of Killing vector fields, expressed in the Cartesian coordinate system,

$$\frac{\partial}{\partial x^0}; \quad x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}, \quad 1 \leq i, j \leq \beta+1; \quad \frac{\partial}{\partial x^A}, \quad A \in \{\beta+2, \dots, D-1\}.$$

Now, we shall turn our attention to solving the differential equation in (3.7).

The above discussed symmetries impose that the metric function G depends only on the distance coordinate r , i.e. $G = G(r)$, as can be verified by explicit analysis of the Killing equations.

If we write the Laplacian acting on G in the form presented in (3.3) and use that $G = G(r)$ together with (4.4), we end up with

$$\Delta G(r) = \frac{1}{r^\beta} (r^\beta G(r)_{,r})_{,r}, \quad (4.5)$$

which in turn implies (3.7) takes the form

$$\frac{1}{r^\beta} (r^\beta G(r)_{,r})_{,r} = - \frac{D-3}{D-2} K c^2 G^{\frac{2}{D-3}} \rho_m, \quad \beta \in \{0, 1, 2, \dots\}, \quad (4.6)$$

where the mass density is given in Table 1.

Let us note the matter distribution can be generalised to a perfect fluid distributed not only on a thin shell, $D = 4$ spherical symmetry is examined e.g. in [14, 15]. Hyperspherical Majumdar-Papapetrou solutions containing thin shells in general number of dimensions (the hypersphericity implies $D = 2 + \beta$) are studied in [16] in a more general setting. It is also possible to study thick (spherical) shells [17].

Table 1: The mass density of the shells \mathcal{M} (4.1) in arbitrary $D \geq 4$

Symmetry	β	Number of shells	Mass density ρ_m/μ
(Hyper)Spherical	$D - 2$	1	$\delta(r - r_0)$
[Unnamed]	$D - 2 > \beta > 1$	1	$\delta(r - r_0)$
Cylindrical	1	1	$\delta(r - r_0)$
Planar	0	2 (parallel)	$\delta(r - r_0) + \delta(r + r_0)$

General form of solution to (4.6) for $|r| \neq r_0$ is

$$G(r) = C_1 + C_0\psi(r), \quad \beta = 1 : \psi = \ln r, \quad \beta \neq 1 : \psi = \frac{|r|^{1-\beta}}{1-\beta}, \quad (4.7)$$

where the absolute value is added to include the planar symmetry case as will be discussed later on.

Of course, interior and exterior values of both the integration constants C_1 and C_0 are different in general. They are determined by matching the interior and exterior (metric) solutions which is done in the following.

We impose continuity of the metric, i.e. the function $G(r)$, across the shell. This condition can be expressed as

$$0 = G|_{\text{int}}^{\text{ext}} = \lim_{\epsilon \rightarrow 0+} [G(r_0 + \epsilon) - G(r_0 - \epsilon)] = C_1|_{\text{int}}^{\text{ext}} + C_0|_{\text{int}}^{\text{ext}} \psi(r_0). \quad (4.8)$$

We have used that $|r| > r_0$ correspond to exterior and $|r| < r_0$ correspond to interior.

Let us note that the constant C_0 is related to the total mass of the shell. Indeed, we may integrate the equation (4.6) multiplied by r^β across the shell. The left hand side yields

$$\lim_{\epsilon \rightarrow 0+} \int_{r_0-\epsilon}^{r_0+\epsilon} (r^\beta G(r))_{,r} dr = [r^\beta G(r)]_{\text{int}}^{\text{ext}} = C_0|_{\text{int}}^{\text{ext}}.$$

The integrated right hand side yields

$$\lim_{\epsilon \rightarrow 0+} \int_{r_0-\epsilon}^{r_0+\epsilon} \frac{D-3}{D-2} K c^2 r^\beta G^{\frac{2}{D-3}} \rho_m dr = \frac{D-3}{D-2} K c^2 r_0^\beta G(r_0)^{\frac{2}{D-3}} \mu,$$

the continuity of metric function G across the shell was used.

Comparing both sides of the integrated equation leads to

$$C_0|_{\text{int}}^{\text{ext}} = -\frac{D-3}{D-2} K c^2 r_0^\beta G(r_0)^{\frac{2}{D-3}} \mu. \quad (4.9)$$

The equations (4.7), (4.8) and (4.9) allow us to write down the function G in a single formula using Heaviside step function H as

$$G(r) = \underbrace{C_1|_{\text{int}} - C_0|_{\text{int}}^{\text{ext}} [\psi(r_0) - \psi(r)]}_{C_{(r)1}} \underbrace{H(r - r_0) + C_0|_{\text{int}} \psi(r)}_{C_{(r)0} \psi(r)}, \quad (4.10)$$

where the term $C_0|_{\text{int}}^{\text{ext}}$ has already been determined and it is given in (4.9).

The Heaviside step function H satisfies three following conditions

$$H(x > 0) = 1, \quad H(x < 0) = 0, \quad \frac{dH(x)}{dx} = \delta(x).$$

In (4.10), we have introduced another step functions $C_{(r)i}$ that may change across the shell but are constant otherwise, i.e.

$$C_{(r)i} = C_{i|\text{int.}} + [C_{i|\text{ext.}} - C_{i|\text{int.}}] H(r - r_0), \quad C_{(r>r_0)i} = C_{i|\text{ext.}}, \quad C_{(r<r_0)i} = C_{i|\text{int.}}.$$

These functions naturally generalise the constants C_1 and C_0 . Because of their quite special dependence on r , we write the r as a lower case index.

With the Einstein equations coupling constant given by $K = \frac{D-2}{D-3} \frac{4\pi\kappa}{c^4}$, see discussion in footnote 2, the equation (4.10) becomes

$$\begin{aligned} G(r) &= C_{1|\text{int}} + C_{0|\text{int}}\psi(r) + \frac{4\pi\kappa\mu}{c^2} r_0^\beta G(r_0)^{\frac{2}{D-3}} [\psi(r_0) - \psi(r)] H(r - r_0) \\ &= C_{(r)1} + C_{(r)0}\psi(r). \end{aligned}$$

Of course, the above result (4.10) holds for one shell only and it followed from examination of the differential equation in (3.7).

Let us consider the $\beta = 0$ case. This time, r can be negative as well and we consider two shells located at $r = \pm r_0$. We assume the space-time possesses a mirror symmetry at $r = 0$, i.e. $G(-r) = G(r)$. Thus we consider G depending on r through $|r|$ only. A careful examination reveals that the relation (4.10) still applies if we replace r by $|r|$.

We can summarize the results obtained thus far as follows

- The metric is

$$ds^2 = G^{-2}(cdt)^2 - G^{2/(D-3)} [dr^2 + r^2 d\Omega_\beta^2 + \delta_{AB} dx^A dx^B], \quad (4.11)$$

where $G = G(r)$ is given in (4.10). In case of $\beta = 0$ replace all r by $|r|$.

- The electrostatic potential $\phi_e(r)$ is related to $G(r)$ according to (3.7). The space-time dependence of the electrostatic potential indicates that $F_{01} = -F_{10}$ are the only non-zero components of the electromagnetic field strength tensor $F_{\alpha\beta}$. These components correspond to electrostatic field along the axis $x^1 = r$. Notice there is no translational symmetry along that axis.
- The mass-charge balance condition (3.8) holds.

Let us note that the number of shells, as long as each shell in the set has the same symmetry, can be generalis/ed to higher values in an obvious manner if centers of the spherical shells, respectively axis of the cylindrical shells, coincide and all the plane shells are parallel.

5 Some examples with attentions to $D = 4$

We shall impose additional conditions on the general solution derived in the preceding section.

1. We assume the potential ϕ_e is constant in the interior (solution) which corresponds well with the non-relativistic results obtained by use of Gauss electrostatic law.

$$\phi_e|_{\text{int}} = \text{const.} \Leftrightarrow G_{,r}|_{\text{int}} = 0 \Leftrightarrow C_{(r<r_0)0} = 0 \Rightarrow G(|r| \leq r_0) = C_{(r<r_0)1}, \quad (5.1)$$

where the “=” (as a subcase of “ \leq ”) in the last relation holds due to continuity of the metric across the shell.

We see that the condition implies $G(|r| < r_0)$ is constant, hence the interior region of space-time is flat and the metric can be transformed into the Minkowski metric by appropriate rescaling of the time and radial coordinates.

Thus it suffices to consider the exterior solution only.

2. If $\beta \geq 2$, then $G(r \rightarrow \infty)$ does not diverge as is clear from (4.7). We naively impose asymptotical flatness condition by requiring

$$\beta \geq 2 : \lim_{r \rightarrow \infty} G(r) = 1 \Rightarrow C_{(r>r_0)1} = 1. \quad (5.2)$$

With the general solution the solution for both $\psi(r)$ and $G(r)$, in equations (4.7) and (4.10), and subsequent fixing of the coupling constant K we obtain

$$\beta \geq 2 : 1 = G(r \rightarrow \infty) = C_{(r>r_0)1} = C_{(r<r_0)1} - \frac{4\pi\kappa\mu}{c^2} r_0^\beta G(r_0)^{\frac{2}{D-3}} \frac{1}{[\beta-1]r_0^{\beta-1}}.$$

The naive asymptotical flatness condition $G(r \rightarrow \infty) = 1$ cannot be satisfied for $\beta \leq 1$ unless the exterior space-time region is flat. Indeed, $\psi(r \rightarrow \infty)$ diverges and hence $|G(r \rightarrow \infty)|$ is finite only if $C_{(|r|>r_0)0} = 0$.

The condition (5.2), applied in case of $\beta \geq 2$, fixes the value of $C_{(r>r_0)1}$. In case of all admissible values of the parameter β , we can suitably rescale the coordinates so that C_1 , either $C_{(r>r_0)1}$ or $C_{(r<r_0)1}$, is fixed again to $1 = C_1$ the same value as implied on the $C_{(r>r_0)1}$ by the condition (5.2).

If $D - 2 - \beta > 0$, then there is at least one coordinate z . The total mass M , as a volume integral of the mass density ρ_m , is infinite. For that reason, we introduce mass per unit "volume" of the z -coordinates, that is finite, and we denote it by $M_{D-2-\beta}$. ($M_0 \equiv M$ is the total mass and it enters the metric function G in the hyper-spherical case, no z -coordinates.)

Then, the *exterior* metric function G can be written using the above introduced mass parameter $M_{D-2-\beta}$ as

$$G = C_{(r>r_0)1} + kM_{D-2-\beta}\psi(r),$$

where k is constant determined by κ , c , D , β and a numerical factor.

Now, we shall perform the rescaling of spatial coordinates r and z , or let us say a (spatial) length scale L .

$$\begin{aligned} L \rightarrow L' = C_1^{-1/(D-3)} L &\Rightarrow M_{D-\beta-2} \rightarrow M'_{D-\beta-2} = C_1^{(D-\beta-2)/(D-3)} M_{D-\beta-2}, \\ G \rightarrow G' &= C_1 G_1, \end{aligned}$$

where we have used the total mass M is unaffected by the rescaling and we have introduced abbreviations $G_1 = G(C_1 = 1)$ and $C_1 = C_{(r>r_0)1}$ for the sake of brevity.

Consequently, the coordinate expression for space-time metric changes as

$$\begin{aligned} ds^2 &= G^{-2}(cdt)^2 - G^{2/(D-3)}(d\vec{x})^2 \\ &\downarrow \\ (ds')^2 &= C_1^{-2}G_1^{-2}(cdt)^2 - C_1^{2/(D-3)}G_1^{2/(D-3)}C_1^{-2/(D-3)}(d\vec{x}')^2. \end{aligned}$$

The C_1 factors at the purely spatial components cancel out so that it suffices to rescale the time coordinate t . Thus the total transformation of coordinates that sets the integration constant C_1 from the exterior metric to 1 is

$$t \rightarrow t' = C_1 t, \quad r \rightarrow r' = C_1^{-1/(D-3)} r, \quad z \rightarrow z' = C_1^{-1/(D-3)} z. \quad (5.3)$$

We drop the primes in the following.

Thus we have transformed away the constant $C_{(|r|>r_0)1}$. Similar rescaling can be performed for $|r| < r_0$ to "remove" $C_{(|r|<r_0)1}$, this is what we do in case of $\beta \in \{0, 1\}$.

By the first condition and continuity of the metric across the shell, (5.1), it follows that the interior and exterior C_1 's are related by linear transformation. Hence the rescaling that sets either the exterior or the interior C_1 equal one removes indeterminacy of the remaining C_1 .

Table 2: Symmetries and adapted coordinate systems in $D = 4$

Symmetry	Shell \mathcal{M}	Coord. system	$(d\vec{x})^2 = \gamma_{ij}dx^i dx^j$
Spherical	\mathbb{S}_2	(ct, r, θ, φ)	$dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\varphi^2$
Cylindrical	$\mathbb{R} \times \mathbb{S}$	(ct, r, φ, z)	$dr^2 + r^2 d\varphi^2 + dz^2$
Planar	\mathbb{R}^2	(ct, r, z_1, z_2)	$dr^2 + dz_1^2 + dz_2^2$

The conditions (5.1) and either (5.2) (in case of $\beta \geq 2$) or the rescaling of coordinates (5.3) (in the interior in case of $\beta \in \{0, 1\}$) are implicitly imposed on all the sub-cases examined in the following.

There are three possible distinct values of β in case of $D = 4$ – the three symmetries summarized in Table 2.

5.1 $D = 4$ spherical shell ($\beta = 2$)

The solution (4.10), subject to both (5.1) and (5.2), combined with expression for total mass (3.9) $M = 4\pi r_0^2 G^2(r_0)\mu$ reduces to

$$ds^2 = \frac{1}{G^2}(cdt)^2 - G^2 [dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\varphi^2],$$

$$G(r \geq r_0) = 1 + \frac{M\kappa}{c^2 r}, \quad G(r \leq r_0) = 1 + \frac{M\kappa}{c^2 r_0}, \quad \frac{M\kappa}{c^2} = \frac{1}{2}R_s \quad (5.4)$$

The exterior solution describes extremal Reissner-Nordström space-time and the constant $\frac{2M\kappa}{c^2}$ in (5.4) can be related to the Schwarzschild radius R_s as given above.

5.2 $D = 4$ cylindrical shell ($\beta = 1$)

Cylindrical space-times were studied e.g. in [4, 18].

Total mass M and charge Q are infinite. Therefore, we introduce corresponding quantities related to a unit length (along the axis) of the cylinder

$$M_1 = M/\text{length} = 2\pi r_0 G^2(r_0)\mu,$$

similarly for the charge. Then we can rewrite the metric function G in terms of M_1 as

$$G(r) = 1 - \frac{2M_1\kappa}{c^2} \ln\left(\frac{r}{r_0}\right) H(r - r_0), \quad (5.5)$$

where line element, in the rescaled coordinates is

$$ds^2 = \frac{1}{G^2}(cdt)^2 - G^2 [dr^2 + r^2 d\varphi^2 + dz^2].$$

5.3 $D = 4$ two parallel plane shells ($\beta = 0$)

We obtain $G(|r| \leq r_0) = C_{(|r| < r_0)1}$ as in the Section 5.2. The constant $C_{(|r| < r_0)1}$ is set to 1 by the coordinate transformation (5.3).

$$G(r) = G(|r|) = 1 + \frac{4\pi\kappa G^2(r_0)\mu}{c^2} [r_0 - |r|] H(|r| - r_0). \quad (5.6)$$

Hence the resulting exterior line element is

$$ds^2 = \frac{1}{G^2}(cdt)^2 - G^2 [dr^2 + dz_1^2 + dz_2^2]$$

with G given by (5.6).

5.4 $D = 5$ hyperspherical shell ($\beta = 3$)

Solution of the $D = 5$ hyperspherical shell with $G(r)$ (for $D = 5$ and $\beta = 3$) subject to both (5.1) and (5.2) is

$$G(r) = 1 + \frac{4\pi\kappa\mu}{c^2} r_0^3 G(r_0) \left(\frac{1}{2r_0^2} + \left[\frac{1}{2r^2} - \frac{1}{2r_0^2} \right] H(|r| - r_0) \right). \quad (5.7)$$

Total mass of the hyperspherical shell is given by $M = 2\pi^2 r_0^3 G(r_0) \mu$. The line element therefore can be written as

$$ds^2 = \frac{1}{G^2}(cdt)^2 - G [dr^2 + r^2 d\theta_1^2 + r^2 \sin^2(\theta_1) d\theta_2^2 + r^2 \sin^2(\theta_1) \sin^2(\theta_2) d\varphi^2],$$

$$G(r \geq r_0) = 1 + \frac{M\kappa}{\pi c^2 r^2}, \quad G(r \leq r_0) = 1 + \frac{M\kappa}{\pi c^2 r_0^2}. \quad (5.8)$$

The exterior solution describes extremal Reissner-Nordström space-time in $D = 5$.

6 $D = 4$ non-relativistic approximation

Non-relativistic, i.e. classical, mechanics of thin shells will be treated in this section.

Let us consider two small pieces of the shell(s) with surface areas dS, dS' , their mutual distance being a . Charge and mass surface densities of the shell(s) are μ and σ .

The pieces mutually interact by means of attractive gravitational and repulsive electrostatic forces.

The total force piece dS acts on the piece dS' is

$$d\vec{F} = \underbrace{-\kappa \frac{\mu^2}{a^2} dS dS' \vec{n}}_{\text{gravitational}} + \underbrace{\frac{\sigma^2}{4\pi\epsilon_0 a^2} dS dS' \vec{n}}_{\text{electrostatic}} = \left[-\kappa\mu^2 + \frac{\sigma^2}{4\pi\epsilon_0} \right] \frac{1}{a^2} dS dS' \vec{n},$$

where \vec{n} is unit length vector pointing from dS to dS' .

If we demand the shell is static, it follows $d\vec{F} = 0$, hence the balance condition $\frac{\sigma^2}{4\pi\epsilon_0} = \kappa\mu^2$. This condition has the same form also for general relativity in four-dimensional space-time.

Let us turn our attention to equation (3.7) which is an example of Poisson equation.

Every $D = 4$ solution of the shells considered in this work can be written in the form $G = 1 - \frac{\mu f(r)}{c^2}$. In the non-relativistic limit, one has $\mu f(r) \ll c^2$ (hence $G \doteq 1$), this may be due to weak gravitational source and / or sufficiently large distance from the source, i.e. low $f(r)$.

If the non-constant term in G is small, then the Poisson equation simplifies to

$$-\Delta[\mu f(r)] = \Delta G c^2 = -\frac{1}{2} K c^4 G^2 \rho_m \doteq -4\pi\kappa\rho_m.$$

This equation is similar to well known Poisson equation for non-relativistic gravitational field.

Let us introduce a new function $\phi_m = \mu f(r) + K_m$. With this definition, the Poisson equation becomes

$$\Delta\phi_m = 4\pi\kappa\rho_m.$$

This is precisely the classical mechanics Poisson equation for the gravitational potential ϕ_m that is determined up to an additional constant K_m .

Concrete examples of gravitational potentials of massive shells can be read off from equations (5.4), (5.5) and (5.6). The potential is in case of a spherical shell

$$\phi_m(r \geq r_0) = -4\pi\kappa \frac{r_0^2 \mu}{r} + K_m, \phi_m(r \leq r_0) = -4\pi\kappa r_0 \mu + K_m, \quad (6.1)$$

in case of a cylindrical shell

$$\phi_m = 4\pi\kappa r_0 \mu \ln\left(\frac{r}{r_0}\right) H(r - r_0) + K_m \quad (6.2)$$

and in case of two planar shells

$$\phi_m = 4\pi\kappa\mu[|r| - r_0]H(|r| - r_0) + K_m. \quad (6.3)$$

This is in agreement with the results of classical mechanics.

A general relation between non-relativistic potential ϕ_m and the metric function G is given by

$$\phi_m = c^2(1 - G) + K_m. \quad (6.4)$$

As a result, the metric coefficients $g_{\alpha\beta}$ of Majumdar–Papapetrou form can be easily obtained if we know the non-relativistic potential.

7 Remark on generalisation to Kastor–Traschen

The Majumdar–Papapetrou space–time can be generalised to a co-moving charged dust and a positive cosmological constant.

Adding a cosmological constant is a motivation to consider a slightly generalised metric ansatz, when compared to (3.1), with G allowed to depend on time.

$$ds^2 = G^{-2}(c dt)^2 - G^{2/(D-3)} \gamma_{ij} dx^i dx^j, \quad G = G(t, x^i). \quad (7.1)$$

The above metric can also be expressed in an alternate form

$$ds^2 = \bar{G}^{-2}(c d\bar{t})^2 - \bar{G}^{2/(D-3)} e^{2a(t)} \gamma_{ij} dx^i dx^j$$

that generalises the $D = 4$ metric ansatz of [6] and it is related to (7.1) by following transformations

$$d\bar{t} = e^{-(D-3)a(t)} dt, \quad \bar{G} = G e^{-(D-3)a(t)}.$$

The difference $\Delta_{\mu\nu} \equiv G_{\mu\nu} - G_{\mu\nu}^{\text{MP}}$ of the Einstein tensor of the above metric (7.1) and the $G_{\mu\nu}^{\text{MP}}$ corresponding to the Majumdar–Papapetrou metric is

$$\Delta_{00} = \frac{\delta_D \epsilon_D}{2} \dot{G}^2 g_{00}, \quad \Delta_{0i} = \delta_D \frac{G_{,0i}}{G}, \quad \Delta_{ij} = \frac{\delta_D}{2} \left[2G\ddot{G} + \epsilon_D \dot{G}^2 \right] g_{ij}, \quad (7.2)$$

where dimension dependent factors

$$\delta_D \equiv \frac{D-2}{D-3}, \quad \epsilon_D \equiv \frac{D-1}{D-3}$$

were introduced for the sake of brevity.

The mixed component of the Einstein equations implies

$$G_{0i} = 0 \Rightarrow G_{,0i} = 0 \Rightarrow G = T(t) + R(x^i). \quad (7.3)$$

The extra terms in G_{00} and G_{ij} , when compared to the case of $\Lambda = 0$, are proportional to metric and hence they can be attributed to the cosmological constant but only for a suitable choice of G given by

$$G = \pm \sqrt{\frac{2\Lambda}{\delta_D \epsilon_D}} t + R(x^i) = \pm \sqrt{\frac{2\Lambda}{(D-1)(D-2)}} (D-3)t + R(x^i). \quad (7.4)$$

The metric function G presented in (7.4) ensures that the relations among $\phi_e(G)$, ρ_e , ρ_m and G (namely the equations (3.7) and (3.8)) have formally the same form as in the case of $\Lambda = 0$, the only difference being that constants of integration with respect to spatial coordinates are functions of time in general.

It is immediate to see that the Laplace operator (of the metric γ_{ij}) acting on G gives $\Delta G = \Delta R$, i.e. R is determined by the classical potential ϕ_m in a similar way as was the pure Majumdar–Papapetrou G , see equation (6.4).

8 Conclusion

We have presented a systematic study of static higher dimensional thin charged dust hypersurface shells of the form $\mathbb{S}^\beta \times \mathbb{R}^{D-2-\beta}$ in arbitrary $D \geq 4$ space–time of the Majumdar–Papapetrou form. The metric is determined by a single function denoted, in this work, by G .

The specific form of the Majumdar–Papapetrou metric reduced the Einstein–Maxwell equations to Laplace equation for the single unknown metric function in the source–free regions. The electrostatic potential ϕ_e was determined by the metric function G and consistency of the equations implied relation between mass and charge densities of the dust.

The Poisson–like relation between G and mass density ρ_m implied the metric function G can be built using a non–relativistic gravitational potential ϕ_m , the relation is presented in equation (6.4).

The obtained results can be further generalised beyond the already addressed Kastor–Traschen space–time. Possible generalisation of the results presented in this article to the case of γ_{ij} or $d\Omega^2$, the former introduced in equation (3.1), being a maximally symmetric metric of arbitrary constant curvature is currently under investigation. It is also possible to study shells that are not hypersurfaces, i.e. $\dim \mathcal{M} < D - 2$, or generalise the results to modified theories of gravity.

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